Algèbre
Matrices – Algèbre des matrices carrées

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Solution 18

$$\begin{pmatrix} 5 & 4 & 3 \\ 7 & 6 & 5 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\iff \left\{ \begin{array}{lll} \lambda_1 + 2\lambda_4 + 3\lambda_7 = 5 & \lambda_2 + 2\lambda_5 + 3\lambda_8 = 4 & \lambda_3 + 2\lambda_6 + 3\lambda_9 = 3 \\ 2\lambda_1 + 3\lambda_4 + 4\lambda_7 = 7 & 2\lambda_2 + 3\lambda_5 + 4\lambda_8 = 6 & 2\lambda_3 + 3\lambda_6 + 4\lambda_9 = 5 \\ \lambda_1 + 2\lambda_4 + 3\lambda_7 = 5 & \lambda_2 + 2\lambda_5 + 3\lambda_8 = 4 & \lambda_3 + 2\lambda_6 + 3\lambda_9 = 3 \end{array} \right.$$

$$\iff \left\{ \begin{array}{lll} \lambda_1 + 2\lambda_4 + 3\lambda_7 = 5 & \lambda_2 + 2\lambda_5 + 3\lambda_8 = 4 & \lambda_3 + 2\lambda_6 + 3\lambda_9 = 3 \\ 2\lambda_1 + 3\lambda_4 + 4\lambda_7 = 7 & 2\lambda_2 + 3\lambda_5 + 4\lambda_8 = 6 & 2\lambda_3 + 3\lambda_6 + 4\lambda_9 = 5 \end{array} \right.$$

$$\iff \left\{ \begin{array}{lll} \lambda_1 + 2\lambda_4 + 3\lambda_7 = 5 & \lambda_2 + 2\lambda_5 + 3\lambda_8 = 4 & \lambda_3 + 2\lambda_6 + 3\lambda_9 = 3 \\ \lambda_4 + 2\lambda_7 = 3 & \lambda_5 + 2\lambda_8 = 2 & \lambda_6 + 2\lambda_9 = 1 \end{array} \right.$$

$$\iff (\lambda_7, \lambda_8, \lambda_9) \in \mathbb{R}^3, \left\{ \begin{array}{lll} \lambda_1 = -1 + \lambda_7 & \lambda_2 = \lambda_8 & \lambda_3 = 1 + \lambda_9 \\ \lambda_4 = 3 - 2\lambda_7 & \lambda_5 = 2 - 2\lambda_8 & \lambda_6 = 1 - 2\lambda_9 \end{array} \right.$$

Solution 20

1. $\jmath = -\frac{1}{2} + \imath\frac{\sqrt{3}}{2}$; $\bar{\jmath} = -\frac{1}{2} - \imath\frac{\sqrt{3}}{2}$; et $1 + \jmath + \bar{\jmath} = 0$.
2. $a + \imath b = a + ((\jmath + \frac{1}{2})\frac{2\sqrt{3}}{3})b = a + \frac{\sqrt{3}}{3}b + \jmath\frac{2\sqrt{3}}{3}b$.

   Et, $z_{\mathcal{B}_2} = \begin{pmatrix} a + \frac{\sqrt{3}}{3}b \\ \frac{2\sqrt{3}}{3}b \end{pmatrix} \in \mathcal{M}_{2,1}(\mathbb{R})$ et $z_{\mathcal{B}_2} = \underbrace{\begin{pmatrix} 1 & \frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{pmatrix}}_{:=A} z_{\mathcal{B}_1}$.
3. $\mu + \jmath\nu = \mu + (-\frac{1}{2} + \imath\frac{\sqrt{3}}{2})\nu = \mu - \frac{1}{2}\nu + \imath\frac{\sqrt{3}}{2}\nu$.

   Et, $z_{\mathcal{B}_1} = \underbrace{\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}}_{:=B} z_{\mathcal{B}_2}$.

   Autre méthode ...

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B est la matrice de passage de la base $\mathcal{B}_1$ vers la base $\mathcal{B}_2$:

$$B = \begin{array}{cc} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} & \begin{array}{c} 1 \\ \imath \end{array} \\ \begin{array}{cc} 1 & \jmath \end{array} & \end{array}$$

A est l'inverse de B. Recherche par opérations élémentaires sur les lignes ...

$$\left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 1 \end{array}\right) \overset{L_1 \leftarrow L_1 + \frac{\sqrt{3}}{3} L_2}{\Longleftrightarrow} \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{2} & 0 & 1 \end{array}\right)$$

$$\overset{L_2 \leftarrow \frac{2\sqrt{3}}{3} L_2}{\Longleftrightarrow} \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{\sqrt{3}}{3} \\ 0 & 1 & 0 & \frac{2\sqrt{3}}{3} \end{array}\right)$$

Et,

$$B^{-1} = \begin{pmatrix} 1 & \frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{pmatrix}.$$

$$C_{\mathcal{B}_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{c} 1 \\ \imath \end{array}$$

$$c(1) = 1 \ c(\imath) = -\imath$$

4.

$$C_{\mathcal{B}_2} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{array}{c} 1 \\ \jmath \end{array}$$

$$c(1) = 1 \ c(\jmath) = -1 - \jmath$$

5. B est la matrice de passage de la base $\mathcal{B}_1$ vers la base $\mathcal{B}_2$ donc $B^{-1} = A$ et $C_{\mathcal{B}_2} = B^{-1} C_{\mathcal{B}_1} B$.